Appendix to
Landscape Dynamics and Conspicuous Consumption *

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Abstract

Proofs and derivations for the paper *Landscape Dynamics and Conspicuous Consumption*, October 2004, for the 2004 Proceedings of the Society for Dynamic Games.

1 Appendix: Proofs and Derivations

1.1 Preliminaries

We begin with some functional analysis, expanding the details given earlier in (Friedman and Yellin, 1997).

Let $A = [0, 1]$, the closed unit interval, and $X = L^1(A, \mathcal{R})$, the Banach space of integrable real-valued functions on A, with the L^1 norm, $||f|| = \int_A f = \int_0^1 f(x)dx$. Note that X is the closure under the L^1 norm of the space of continuous functions, $\mathcal{C}(A, \mathcal{R})$.

The space of measures is $X' = L(X, \mathcal{R})$, the Banach space dual to X. The usual notation for measure is μ, with the linear functional on X written as $g \mapsto \int_A g(y)d\mu(y)$ (Rudin, 1991). An alternate notation, more convenient for our purposes, is to represent each element of X' by a cumulative distribution function F, where $F(x) = \mu[\{y \in A : y \leq x\}]$ and the map above is rewritten using the Stieltjes integral as $g \mapsto \int_A g(y)dF(y)$. Standard texts (Billingsley, 1995) show that the set of probability measures on A coincides with the set of right-continuous, nondecreasing functions F on A with $F(0) = 0$ and $F(1) = 1$.

The appropriate topology on X' involves reference to its dual space X''; see (Lax, 2002; p. 118) for details. A function $g \in X$ induces a form $g'' \in X''$ defined by the Stieljes integral,

$$g'' : X' \to \mathcal{R} : F \mapsto \int_A g(y)dF(y)$$

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Let X^{**} denote the image of X in X'' under this map. The *weak-star topology* is the weakest topology on X' in which every such g'' is continuous. The closed unit ball in X' is compact in this topology, a property we will find useful below.

Recall that for the Envy specification of Veblen consumption, the master equation is

$$F_t = F_x[F - (c/x)]. \tag{1}$$

where the subscripts denote partial derivatives. Also recall that the pure atomic measure concentrated at a point $x_o \in A$ is represented by the cumulative distribution function $\Theta(x - x_o)$ on A, where the unit step (or Heaviside) function $\Theta(y) = 0$ if $y < 0$ and $= 1$ if $y \geq 0$.

1.2 Proof of Proposition 1

Proposition 1. Let the distribution $F(x,t)$ be a solution to (1) for a given initial condition $F_o(x) = F(x, 0)$, and let $\widetilde{x} = \sup\{x \in [0,1] : xF_o(x) < c\}$. Then as $t \to \infty$, $F(x,t)$ converges pointwise to $\Theta(x- \widetilde{x})$.

Lemma. For all $x \in [0,1]$ and all $t \geq 0$, we have either $x = \widetilde{x}$ or $sgn\,[F(x,t) - c/x] = sgn\,[x - \widetilde{x}]$, where $sgn(z) = -1, 0$, or 1 indicates whether z is negative, zero or positive.

Proof of Lemma. Fix $0 < c < 1$ and assume for the moment that $F(x,t)$ is a smooth function of x for all $t \geq 0$. Let $x = \widetilde{x} \in (0,1)$ solve $F(x,0) = c/x$. The solution $x = \widetilde{x}$ is unique and interior because the function $F(x) - c/x$ (= -gradient of Veblen Pride utility) is continuous, strictly increasing, negative at $x = 0$ and positive at $x = 1$. Hence the above sign relation holds at time $t = 0$. Furthermore, no mass moves past the point $x = \widetilde{x}$ at $t = 0$ because the velocity field at $x = \widetilde{x}$ is $F(\widetilde{x}, 0) - c/\widetilde{x} = 0$. The same flow restriction holds at later times by the same argument. Thus $F(\widetilde{x}, t) = F(\widetilde{x})$ holds in $t \geq 0$, and the conclusion holds for all $t \geq 0$.

In the case in which $F(x,t)$ is discontinuous in x, define $\widetilde{x}$ as in the Proposition. The previous argument is unchanged by discontinuities away from $\widetilde{x}$, so suppose that there is a discontinuity at $x = \widetilde{x}$ for some $t \geq 0$. For small $\epsilon > 0$ the same argument shows that $F(\widetilde{x} - \epsilon, t)$ is decreasing and that $F(\widetilde{x} + \epsilon, t)$ is increasing in t at all points of continuity. Thus the expressions don't change sign and the conclusion still holds. Again, the point $x = \widetilde{x}$ absorbs mass from both directions. Finally, for the case $c = 0$, it is straightforward to verify that the Lemma holds with $\widetilde{x} = 0$.■

Proof of Proposition. It suffices to construct a Ljapunov function V and verify: (a) as a functional on the space of cdf's on $[0,1]$, V attains a global minimum at F^*, the unit step function at $\widetilde{x}$; and (b) as a function of time along a solution $F(x,t)$, the value of V is strictly decreasing at all times when $F(x,t) \neq F^*$. To verify (a), let $V(t) = \int_0^1 (x - \widetilde{x})^2 dF(x,t)$. The integrand is zero at $x = \widetilde{x}$ and where $F(x,t)$ is locally constant. Elsewhere the integrand is positive, and thus (a) is verified. To verify (b) integrate by parts to obtain $V(t) = (1 - \widetilde{x})^2 - 2\int_0^1 (x - \widetilde{x}) F(x,t) dx$. Hence the time derivative exists and is equal to $\dot{V} = -2\int_0^1 (x - \widetilde{x}) F_t(x,t) dx$. Using (1), we have $\dot{V} = -2\int_0^1 (x - \widetilde{x})[F(x,t) - c/x] dF(x,t)$. The Lemma now tells us that $\dot{V}$ is negative except at $F(x,t) = F^*$, and (b) follows. ■

1.3 Derivation of General Solution to (1)

Write (1) in the form

$$F_t - [F - (c/x)]F_x = 0. \tag{2}$$

A solution $F(x,t)$ of (2) defines a surface in $[F(x,t),x,t]$ space.[1] Along any such integral surface the total time derivative is

$$dF(x,t)/dt = F_t + F_x dx/dt. \tag{3}$$

Comparing (2) and (3), we see that (2) defines a set of curves along each of which

$$\begin{aligned} dF/dt &= 0, && (4)\\ dx/dt &= c/x - D. && (5) \end{aligned}$$

Equation (5) describes the particular time path, or characteristic curve, $x(t)$ followed by a consumer with position $x(0)$ at time $t = 0$. Equation (4) tells us that along such a characteristic curve, $F(x,t) =$ constant.

To characterize solutions of (4, 5), it is useful to define an auxiliary variable $z = z(x,t)$ implicitly given by $F(x,t) = F_o(z)$. Because F is constant along each characteristic curve defined by (4, 5), we label each such curve by the corresponding value of z. Separating variables and substituting $F_o(z)$ for F, we have $dt = xdx/(c - xF_o(z))$. Because z is fixed along any characteristic, this expression can be integrated directly using the textbook formula

$$\frac{x}{1-ax} = \frac{1}{a^2}\frac{d}{dx}[1 - ax - \ln(1-ax)].$$

We obtain

$$t + t_0(z) = \frac{1 - xF(z)/c - \ln|1 - xF(z)/c|}{F^2(z)/c}. \tag{6}$$

In (6), the integration constant $t_0(z)$ is constant along each characteristic but varies across characteristics. By the definition of z, $x = z$ at $t = 0$. Hence

$$t_0(z) = \frac{1 - zF(z)/c - \ln|1 - zF(z)/c|}{F^2(z)/c}, \tag{7}$$

and we may subtract (7) from (6) to obtain the implicit solution

$$t = \frac{z-x}{F(z)} + \frac{c}{F^2(z)}\ln\left(\frac{|c - zF(z)|}{|c - xF(z)|}\right) \tag{8}$$

of the master equation (see equation (5) of the paper).

We now derive the behavior over time of the shock position and magnitude, given an initial probability density $f(x)$ and corresponding cumulative distribution $F_o(x)$. The $c = 0$, pure rank dependent consumption limit provides an analytic solution while preserving the qualitative behavior of the shock wave.

[1]The authors thank Joel Yellin for providing the key ideas in this derivation.

In general, three conditions determine the shock position $s(t)$ and the leading and trailing values z_L,z_R that mark the left and right edges of the shock in terms of the auxiliary variable z.[2] The first two conditions apply the solution (8) of the underlying dynamic equation to the leading and trailing edges of the shock, viz.

$$t = \frac{z_L - s}{F(z_L)} + \frac{c}{F^2(z_L)} \ln\left(\frac{c - z_L F(z_L)}{c - sF(z_L)}\right); \tag{9}$$

$$t = \frac{z_R - s}{F(z_R)} + \frac{c}{F^2(z_R)} \ln\left(\frac{c - z_R F(z_R)}{c - sF(z_R)}\right), \tag{10}$$

The third condition comes from applying the integral form of the underlying conservation law across the shock, using the weak-star topology to obtain the desired limiting expressions. Substitution of $\phi_y(y, F) = \frac{c}{y} - F(y, t)$ into a general expression for conservation of mass yields

$$-\frac{\partial}{\partial t}\int_{x_1}^{x_2} F(y,t)dy = -\int_{F(x_1,t)}^{F(x_2,t)} \left[\frac{c}{y} - F(y,t)\right] dF(y,t). \tag{11}$$

Suppose $F(x, t)$ has a jump discontinuity at $x = s(t)$, and choose $x_1 < s(t) < x_2$. Then in the limit as $x_1 \to s(t)$ from below and $x_2 \to s(t)$ from above, (11) becomes

$$-\left[F(z_R) - F(z_L)\right]\frac{ds}{dt} = -\frac{c\left[F(z_R) - F(z_L)\right]}{s} + \frac{1}{2}\left[F^2(z_R) - F^2(z_L)\right].$$

On division by the shock magnitude $F(z_R) - F(z_L)$, we obtain the shock velocity

$$\frac{ds}{dt} = -\frac{1}{2}[F(z_L) + F(z_R)] + \frac{c}{s}. \tag{12}$$

For $c = 0$, (9) and (10) can be combined and expressed in the symmetric forms

$$s = \frac{1}{2}\left[z_L + z_R\right] - \frac{t}{2}[F(z_L) + F(z_R)], \tag{13}$$

$$t = \frac{z_R - z_L}{F(z_R) - F(z_L)}. \tag{14}$$

Furthermore, on differentiating (13) with respect to t, equating the result with the $c = 0$ form of (12), and substituting for t from (14), we obtain

$$\left[z_R - z_L\right]\left[F'(z_L)\dot{z}_L + F'(z_R)\dot{z}_R\right] = \left[F(z_R) - F(z_L)\right]\left[\dot{z}_L + \dot{z}_R\right], \tag{15}$$

which integrates to the "equal area condition"

$$\frac{1}{2}[F(z_L) + F(z_R)]\left[z_R - z_L\right] = \int_{z_L}^{z_R} F(z)dz. \tag{16}$$

[2]These are the Rankine-Hugoniot conditions. The procedure used here to derive the equal area condition (16) below is laid out in Whitham (1974). See Smoller (1994) for formal discussion. To our knowledge, the derivation of the shock initiation time and magnitude, given initial symmetric beta density, is new.

Equation (16) states that the shock cuts off equal areas from the breaking wavefront, preserving population mass. This interpretation becomes transparent if we rewrite (16) in the form

$$\int_{z_L}^{\widehat{z}} \left\{ \frac{1}{2}[F(z_R) - F(z_L)] - F(z) + F(z_L) \right\} dz = \int_{\widehat{z}}^{z_R} \left\{ F(z) - F(z_L) - \frac{1}{2}[F(z_L) - F(z_R)] \right\} dz,$$

which explicitly equates the areas swept out by the right and left "lobes" of the shock. Note that the crossover point of the shock, $z = \widehat{z}$, is determined by $F(\widehat{z}) = \frac{1}{2}[F(z_L) + F(z_R)]$.

To determine the shock initiation time t^*, it is helpful to write (14) in terms of the density $f(z)$. Introduce $\Delta = [z_R - z_L]/2$, $\overline{z} = [z_R + z_L]/2$, so that $\Delta(t^*) = 0$ when a shock initiates. Then (14) becomes

$$\frac{1}{2\Delta} \int_{-\Delta}^{\Delta} f(z + \overline{z}) dz = \frac{1}{t}, \tag{17}$$

and in the limit $\Delta \to 0$ we have

$$\frac{1}{t^*} = f(z). \tag{18}$$

Certain basic relations follow from the assumed unimodality and symmetry about $z = 1/2$ of the density $f(z)$. Symmetry $f(z + 1/2) = f(-z + 1/2)$ gives $\overline{z} = 1/2$,

$$\int_0^{1/2-\Delta} f(z) dz = \int_{1/2+\Delta}^{1} f(z) dz,$$

and therefore

$$F(z_L) + F(z_R) = F(1/2 - \Delta) + F(1/2 + \Delta) = 1. \tag{19}$$

¿From $\overline{z} = 1/2$, (13) and (19), the position of shock at time t is

$$s(t) = \frac{1}{2}(1 - t),\ t > t^*, \tag{20}$$

where from (18), the shock initiation time satisfies

$$\frac{1}{t^*} = f(1/2). \tag{21}$$

In the symmetric $c = 0$ case, the shock therefore reaches position $s = x = 0$ at $t = 1$.

The shock initiation time and magnitude now follow for the special case described in the text. This solution is one instance from an infinite class of analytic, $c = 0$ shock solutions for the conservation law (1), for the family of initial unimodal beta densities symmetric about $z = 1/2$,

$$f_a(z) = \frac{z^{a-1}(1-z)^{a-1}}{B(a, a)},\ a > 1, \tag{22}$$

with corresponding distributions $F_a(z)$, where the beta function $B(a, a) = [(a-1)!]^2/(2a-1)!$.

Equations (21, 22) give the shock initiation time

$$t^* = \frac{2^{1-2a}}{B(a,a)}. \tag{23}$$

¿From (14), the shock magnitude is given implicitly by

$$2F_a(1/2 + \Delta) - 1 = \frac{2\Delta}{t}. \tag{24}$$

For the beta density $a = 2$ considered in the text, $F_2(z) = z^2(3 - 2z)$, and (24) reduces to a quadratic equation in Δ,

$$\frac{1}{t} = \frac{3}{2} - 2\Delta^2. \tag{25}$$

From (25) or (21), the shock initiates at $t^* = 2/3$. Equation (20) gives the initial shock position $x^* = 1/6$, as stated in the text. The shock magnitude thereafter is given by (25).

1.4 Proof of Proposition 2

Proposition 2. Let the initial distribution $F(x)$ be thrice continuously differentiable, with a regular strict maximum at $x = q \in (0,1)$. Then for all sufficiently small $c > 0$, the solution of (1) has a moving interior shock. Up to first order in c, the shock emerges at time

$$t^*(c) = 1/f(q) + cT(q) + O(c^2)$$

and location

$$x^*(c) = q - F(q)/f(q) + cX(q) + O(c^2) \in (0,1).$$

Here,

$$T(q) = \left[\frac{\overline{\gamma}_z}{f(q)} - \frac{\overline{\gamma}_{zz}}{f''(q)} + \frac{\overline{\gamma}_z f'(q)}{f(q)f''(q)}\right];$$

$$X(q) = \left[\overline{\gamma} - \frac{\overline{\gamma}_z F(q)}{f(q)} + \frac{\overline{\gamma}\overline{\gamma}_{zz} F(q)}{f''(q)} - \frac{\overline{\gamma}_z f'(q) F(q)}{f(q)f''(q)}\right],$$

where, defining $Q = \ln \frac{qf(q)}{qf(q)-F(q)}$,

$$\overline{\gamma} = \frac{Q}{F(q)};$$

$$\overline{\gamma}_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)};$$

$$\overline{\gamma}_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{q^2 f'(q)Q + 2qf(q) + F(q)}{q^2F^2(q)}.$$

Proof. Recall that the general implicit solution to the initial value problem is $0 = \alpha_1 \equiv tF(z) - z + x - c\gamma$, where

$$\gamma(c, z, x) \equiv \frac{1}{F(z)} \ln \frac{|zF(z) - c|}{|xF(z) - c|},$$

z is the auxiliary variable defined by $F(z) = F(x, t)$, and we have made the dependence on c explicit. Recall further that any boundary point (x, t) of a shock is a singularity. That is, we have $0 = \alpha_2 \equiv 1 - tf(z) + c\gamma_z$, where $f = F'$ is the initial density. We seek the earliest time where a singularity occurs, i.e., a minimal positive value of t in the equation $0 = \alpha_2$. The associated first order condition is $0 = \alpha_3 \equiv cf(z)\gamma_{zz} - (c\gamma_z + 1)f'(z)$.

For fixed $c \geq 0$, let $\Phi(c, \cdot, \cdot, \cdot) : [0, 1]^2 \times \mathbf{R}_+ \to \mathbf{R}^3$ be the function that maps the point (z, x, t) to $(\alpha_1, \alpha_2, \alpha_3)$. Consider first the case $c = 0$. Here $0 = \alpha_3$ implies $f'(z) = 0$, so the shock first appears on the characteristic curve associated with the interior maximum $z = q$. From $0 = \alpha_2$ we infer that the shock initiates at time $t^*(0) = 1/f(q)$, and from $0 = \alpha_1$ we infer that the initial shock location is $x^*(0) = z - t^*F(z) = q - F(q)/f(q)$. It is clear that $0 < x^*(0) < q < 1$. We have $x^*(0) > 0$ because, at the global maximum $q > 0$ of $f(q)$, $F(q) = \int_0^q f(y)dy < qf(q)$. Thus we have an interior shock emerging in finite time for $c = 0$.

For small positive c, we apply the implicit function theorem to $\Phi(c, \cdot, \cdot, \cdot)$. The key condition (see e.g. Spivak, 1965) is that the Jacobian determinant $|\mathbf{J}(\alpha_1, \alpha_2, \alpha_3; z, x, t)|$is not zero when evaluated at the point $c = 0$, $z = q$, $x = q - F(q)/f(q)$, and $t = 1/f(q)$. Explicitly,

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial\alpha_1}{\partial z} & \frac{\partial\alpha_1}{\partial x} & \frac{\partial\alpha_1}{\partial t} \\ \frac{\partial\alpha_2}{\partial z} & \frac{\partial\alpha_2}{\partial x} & \frac{\partial\alpha_2}{\partial t} \\ \frac{\partial\alpha_3}{\partial z} & \frac{\partial\alpha_3}{\partial x} & \frac{\partial\alpha_3}{\partial t} \end{vmatrix} = \begin{vmatrix} tf(z) - 1 & 1 & F(q) \\ tf'(q) & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & F(q) \\ 1 & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = -f(q)f''(q).$$

The Jacobian is strictly positive because the density is positive at any maximum and has a negative second derivative at a regular maximum. Hence the desired implicit functions exist and have derivatives (evaluated at the same point) given by

$$\begin{pmatrix} z^{*\prime}(0) \\ x^{*\prime}(0) \\ t^{*\prime}(0) \end{pmatrix} = -\mathbf{J}^{-1} \begin{pmatrix} \frac{\partial\alpha_1}{\partial c} \\ \frac{\partial\alpha_2}{\partial c} \\ \frac{\partial\alpha_3}{\partial c} \end{pmatrix} = - \begin{pmatrix} 0 & 0 & \frac{1}{f''(q)} \\ 1 & \frac{F(q)}{f(q)} & \frac{-F(q)}{f(q)f''(q)} \\ 0 & \frac{-1}{f(q)} & \frac{1}{f(q)f''(q)} \end{pmatrix} \begin{pmatrix} -\gamma \\ \gamma_z \\ f(q)\gamma_{zz} - \gamma_z f'(q) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f(q)\gamma_{zz} - \gamma_z f'(q)}{f''(q)} \\ \gamma - \frac{\gamma_z F(q)}{f(q)} + \frac{\gamma_{zz}F(q)}{f''(q)} - \frac{\gamma_z f'(q)F(q)}{f(q)f''(q)} \\ \frac{\gamma_z}{f(q)} - \frac{\gamma_{zz}}{f''(q)} + \frac{\gamma_z f'(q)}{f(q)f''(q)} \end{pmatrix}.$$

The values of γ and its derivatives are readily calculated at $c = 0$. Using the notation $Q = \ln |z| / |x| = \ln \frac{qf(q)}{qf(q) - F(q)}$, we find that at the relevant point [$z = q$; $x = q - F(q)/f(q)$; $t = 1/f(q)$] we have

$$\gamma = \frac{Q}{F(q)},$$

$$\gamma_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)},$$

and

$$\gamma_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{q^2 f'(q)Q + 2qf(q) + F(q)}{q^2 F^2(q)}.$$

The expressions in the Proposition now follow from the first order Taylor expansion at $c = 0$. They are valid as long as the shock position $x^*(c)$ remains above the zero $\widetilde{x}(c)$ of the gradient ϕ_x corresponding to the condition $x/c = F(x)$. Clearly $\widetilde{x}(0) = 0$, and $\widetilde{x}(c)$ is continuous in c because F has a density, so the condition $x^*(c) > \widetilde{x}(c)$ holds for sufficiently small c. ■

1.5 References

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