CASTLES IN TUSCANY: THE DYNAMICS OF RANK DEPENDENT CONSUMPTION

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Abstract. We apply gradient dynamics to population games in which consumers interact via rank dependent preferences. Rank dependent consumption leads to a payoff landscape that changes shape in response to the nonlinearity inherent in non-price interactions. The nonlinearity causes the spontaneous formation of moving accreting clusters in action space. We study the structure and development of these clusters in deterministic models and also in the presence of stochastic uncertainty about the shape of the local landscape. The full model thus combines linear diffusion with nonlinear non-price signaling. Similar linear diffusion has been described in natural resource pricing, and analogous nonlinear saturation due to externalities has been used to study market penetration of pharmaceuticals and other products. It appears possible to extend the same ideas to financial markets. We discuss extensions that may shed light on the dynamics of asset price bubbles.

KEYWORDS: Gradient dynamics, non-price signaling, Brownian noise, population games, payoff landscapes, rank dependent preferences, herding.

"Since consumption of these more excellent goods is an evidence of wealth, it becomes honorific; and conversely, the failure to consume in due quantity and quality becomes a mark of inferiority and demerit." –Thorstein Veblen (1899, p.64)

1. Introduction

Economic theory relies on the price mediated effects of independent individual decisions. Nevertheless, in crucial periods markets often appear to be driven by non-price interactions: bandwagon behavior, herding, and other collective behaviors that manifest interdependence.[1] Consumption decisions are particularly susceptible

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[1]Herding and other mimetic market behaviors have been studied extensively, see, e.g., Bikhchandani et al (1992), Lee (1998), Bannerjee (1992), Topol(1991), Trueman(1994). This previous work falls roughly into two classes, one based on the flow of information to actors with rational expectations, e.g. Avery and Zemsky (1998), the other on mimetic behavior following from mechanical rules, e.g. Kirman (1993). Considered as a description of certain types of herding, our models are rule based. However, we depart from the literature in several respects. We introduce nonlinear self-consistency conditions that determine decision probabilities endogenously, rather than externally fixing all model parameters. Furthermore, our system states are defined by entire probability distributions on a metric that measures allocation of expenditures to rank dependent and ordinary consumption types. This results in models that are in some respects broader and in some respects narrower than previously considered. In particular, our dynamics are Markovian so only current information about decisions affects next-instant decisions. In contrast to Avery and Zemsky (1998), we do not consider an entire decisionmaking history. On the other hand, our Markovian dynamics operates on a multiply infinite functional state space, rather than on a finite state system (as in Kirman (1993)), so that we are able to study non-equilibrium distributive behavior.

to linkage of this sort when they flow from the desire for rank and status.[2] Rank dependent consumption by definition achieves the desired effect only to the extent that it equals or exceeds the consumption of others. Thus, the creation of non-pecuniary negative externalities with dynamic consequences not yet understood.

We take the view here that rank dependent decisions create a promising test bed for dynamic modelling in which agents react to cues other than price. We assume dynamics that operate on a continuum of consumers, each of whom chooses from a continuous action set mapped to $[0,1]$. An action choice specifies the fraction of income devoted to rank dependent consumption. Each action yields a payoff that depends on the current distribution of actions across the population. Consumers adjust their choices continuously so as to move uphill in a payoff "landscape." These individual adjustments change the action distribution and in turn modify the landscape. The interplay of population distribution and landscape characterize an inherent nonlinearity in the dynamics. The full consequences of the nonlinearity emerge when agents are uncertain about the shape of the local topography, and there is a random element in individual adjustments. In its complete form, the model combines linear diffusion and nonlinear saturation effects. Similar linear dynamic diffusion has been described in natural resource pricing,[3] and phenomenological models based on logistic saturation have separately been used to study market saturation of pharmaceuticals and other products.[4]

We examine two rules for rank dependent preferences that produce two distinct dynamic patterns. Under the first rule, individuals are rewarded proportionately as they narrow the gap between their consumption and higher consumption levels of others. The result is a consumption pattern that converges to a homogeneous, single-group equilibrium in which there is a fixed proportion of rank dependent and ordinary consumption. Starting from an initially dispersed behavior pattern, moving subgroups of agents emerge and snowball, continuously attracting more population. Accreting subgroups appear to be a defining feature of an infinite class of models with non-price interdependence. The second rule proportionately rewards consumption in excess of others' consumption. This leads to the emergence of a specific dispersed consumption pattern, even when the initial pattern is homogeneous. In the ultimate equilibrium, a large subgroup of the population

[2]Notions of conspicuous consumption and pecuniary taste, commonly attributed to Thorstein Veblen but anticipated in important ways by John Rea (1834) and others, continue to echo in the modern literature. Robert Frank (1985, p.7) has called attention to the class of "goods that are sought after less because of any absolute property they possess than because they compare favorably to others in their class." See also Loch et al (1999), J. Duesenberry (1949), F. Hirsch (1976).

[3]Dixit and Pindyck (1994, pp.150ff, 404 ff) describe a mean reverting, stochastic process for the value of a developed oil reserve. The dynamics are based on a risk neutral portfolio incorporating a long position in the option to invest in exploration and a short position in the underlying asset. The two competing effects result in an equilibrium state governed by a confluent hypergeometric function. Below in Section 6, we formulate dynamic equations describing the interplay between stochastic uncertainty about the local shape of the payoff landscape and the payoff to ordinary consumption. The linear part of the equilibrium of this system is also described by a confluent hypergeometric function. The dynamics of the linear part of our model and of the resource pricing example considered by Dixit and Pindyck are solvable in terms of similar Green's functions. We conjecture, but have not been able to prove, that the linear part of our model can be expressed as a mean reverting process, as in the resource pricing example. The open question is whether the confluent singularity structure is generic to an interesting class of stochastic economic models.

[4]See Berndt, Pindyck, and Azoulay (1999, 2000) and references cited there.

exhibits a preference pattern dominated by rank dependent consumption, while a small subgroup exhibits a preference for ordinary consumption.

The discussion is organized as follows. Section 2 specifies the payoff functions following from the two different consumer preference rules. Section 3 derives dynamic equations that describe the evolution of the population distribution under the two preference rules. Section 4 gives analytic results for steady state consumption patterns and exhibits general dynamical solutions. Section 5 describes markers for population clusters in the form of shock waves in action space. Section 6 describes the asymptotic dynamic equilibria that emerge when fixed proportions of the population follow the two preference rules simultaneously and there is uncertainty regarding the shape of the local topography. Section 7 summarizes the work, describes the larger universe of models of this general type, and suggests possibilities for extensions. An appendix details proofs and derivations.

2. The Preference Landscape

We consider a population in which at each instant of time, t, individuals choose an action x from the interval $[0,1]$. We interpret an action as an allocation of income in which a fraction x is allocated to ordinary consumption and a fraction $1-x$ is allocated to rank dependent consumption. Individuals are distinguished only by their action choice. Here, income is an endowment with unit flow rate and is instantaneously consumed.

In this context, we introduce a payoff (or overall utility) function ϕ in the form of two additive components. The first is a utility $u(x)$ of ordinary consumption of food and other necessities. The second is a utility U of rank dependent consumption. The utility U has a functional dependence on the distribution of agents' current action choices $D(x,t)$. We assume a sufficiently large population that the set of individuals forms a continuum, so that $D(x,t)$ is piecewise continuous in x. Thus, there is a tradeoff between rank dependent and ordinary consumption,[5] and each consumer has a payoff of the separable form

$$\phi = U + cu, \tag{1}$$

where $c \geq 0$ is the constant marginal rate of substitution. The preference landscape is not fixed. Because the payoff for rank dependent consumption depends on the distribution $D(x,t)$, the dynamical equations described below in Sections 3 and 6 ultimately determine the shape of the preference landscape.

2.0.1. **Ordinary consumption.** We assume that an individual receives direct utility $cu = c\ln x$ from ordinary consumption x. For $c > 0$ the unbounded growth of marginal utility $\partial u/\partial x$ as $x \to 0$ insures that ordinary consumption is a necessity. The dynamics of the model do not depend on the specific choice $u = \ln x$. The same qualitative results follow from any (concave) utility function in the family cx^k, $0 < k < 1$, with diminishing marginal utility and infinite marginal utility at $x = 0$. To highlight our results on emergent homogeneity and heterogeneity, we shall assume that every consumer has the same tradeoff parameter c, and the same fixed overall income, normalized to 1. Hence a consumer with ordinary consumption level x in $[0,1]$ devotes $1-x$ to rank dependent consumption.

[5] We do not consider the choice of particular items within either the rank dependent or ordinary consumption bundle. Presumably, expenditures are allocated efficiently within each bundle, so it suffices to model expenditure shares across the two bundles.

2.1. **Emulation.** We consider two different reward patterns for rank dependent consumption. Under the Emulation reward pattern, an individual receives disutility to the degree that his or her rank dependent consumption is exceeded by the rank dependent consumption of others. Thus an individual choosing the fraction of rank dependent consumption $1-x$, when all others choose $1-y$ in $[0,1]$, receives disutility proportional to the amount by which $1-y$ exceeds $1-x$, viz.

$$r_E(x,y) = \min\{0, (1-x) - (1-y)\} = \min\{0, y-x\}. \tag{2}$$

If others' choices of ordinary consumption y are distributed according to the cumulative distribution function $D(y)$, then from (2) the payoff is

$$U^E(x,D) = \int_0^1 r_E(x,y)dD(y) = \int_0^x (y-x)dD(y). \tag{3}$$

Integrating by parts, we can write (3) as

$$U^E(x,D) = -\int_0^x D(y)dy. \tag{4}$$

Thus, given the population distribution of consumption D, an individual choosing ordinary consumption level x receives total payoff

$$\phi^E(x,D) = c\ln x - \int_0^x D(y)dy. \tag{5}$$

The smaller the ordinary consumption x of a given consumer, holding the consumption of all other consumers fixed, the lower the return from ordinary consumption represented by $c\ln x$, and the higher the return from rank dependent consumption represented by $-\int_0^x D(y)dy$. For sufficiently small x relative to the ordinary consumption of others, $\int_0^x D(y)dy = 0$, which is the maximum possible return to rank dependent consumption under Emulation. Furthermore, because $\int_0^x D(y)dy \leq 1$, whereas $u = \ln x$ is unbounded below as x approaches zero, no individual completely neglects ordinary consumption except in the limiting case $c = 0$.[6]

2.2. **Excess.** An alternative assumption is that an individual receives additional utility ("Excess") based on how much his or her rank dependent consumption exceeds the rank dependent consumption of others. Suppose a particular individual chooses rank dependent consumption $1-x$ when all others choose $1-y$. Under the Excess regime, this individual receives

$$r_X(x,y) = \max\{0, (1-x) - (1-y)\} = \max\{0, y-x\}. \tag{6}$$

If others' ordinary consumption choices y are distributed according to the cumulative distribution function $D(y)$, then from (6) the corresponding total payoff is

$$U^X = \int_0^1 r_x(x,y)dD(y) = \int_x^1 (y-x)dD(y). \tag{7}$$

[6] The case $c \geq 1$ is uninteresting because it implies that the rank dependent term is dominated by the ordinary consumption term in (5), and increasing ordinary consumption always increases the payoff.

Integrating by parts, we can rewrite (7) as

$$U^X = \langle x \rangle - \int_0^x [1 - D(y)]dy, \tag{8}$$

where $\langle x \rangle$ is the population-mean choice

$$\langle x \rangle = \int_0^1 x dD(x) = \int_0^1 [1 - D(x)]\, dx.$$

Thus the individual's overall payoff in the Excess version of rank dependent consumption is

$$\phi^X(x, D) = c \ln x + \langle x \rangle - \int_0^x [1 - D(y)]dy. \tag{9}$$

Under the Excess rules, as under Emulation, no one completely neglects ordinary consumption except in the limiting case $c = 0$. Note that the gradient of the Excess payoff is precisely the Emulation payoff gradient from (5) with the distribution $D(x)$ replaced by the survival function $1 - D(x)$.[7]

3. Gradient Dynamics in Action Space

Given population and action continua and unique labeling of each consumer by an allocation x, it follows that the state of the economy at time $t \in [0, \infty)$ can be described by a probability density $\rho(x, t)$ in action space, or equivalently by a cumulative distribution function $D(x, t) = \int_0^x \rho(y, t)dy$.[8] We assume that each consumer adjusts his or her action choices continuously, so that the state is a continuous function of time.[9]

Following the gradient selection dynamics in organismic biology[10] and the analogous gradient descent algorithms in machine learning,[11] we assume that the time

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[7]Mixed Excess and Emulation dynamics. An alternative dynamics obtains if we fix the relative weights of Emulation and Excess in the preference function via $r(x, y) = a \max(0, y - x) + b \min(0, y - x)$, with $b > a \geq 0$ (or $a > b \geq 0$). Corresponding to the two inequalities, there are two forms of equilibrium, one in which the asymptotic density is an atom, as in the pure Emulation model, the other in which the asymptotic density results in a dispersed distribution, as in the pure Excess model. The first case, $b > a \geq 0$, is consistent with "loss aversion," as documented in the behavioral economics literature (e.g., Rabin 1998, p.14). Note also that with equal weights $a = b = 1/2$ the dynamical equations linearize, and the time dependent general solution of the Cauchy problem can be expressed as a bilinear form (Green's function) in confluent hypergeometric functions. In Section 6 we consider a dynamically more interesting and complex extension, with interacting Excess and Emulation subgroups.

[8]The state space is the set of probability measures on $[0, 1]$ endowed with the usual weak-star topology. It is well known that every such measure can be represented by a distribution function, i.e., the state at given time t is a function $D(x, t)$ that is nondecreasing in x, with $D(x, t) = 0$ for $x < 0$ and $D(x, t) = 1$ for $x \geq 1$. A probability density function is a non-negative function, zero outside of $[0, 1]$, that integrates to 1. The set of densities is an equivalent representation if we include the improper densities known as Dirac delta functions.

[9]Our approach is consistent with stochastic approximation theory, e.g., Ljung and Soderstrom (1983), Benaim and Hirsch (1996), who show that key stability properties of discrete time stochastic dynamics are captured by a continuous time formalism. See Binmore and Samuelson (1994) for other reasons to focus on continuous time deterministic dynamics.

[10]See, e.g., Lande, 1982.

[11]See, e.g., Bishop, 1995, sec. 7.5.

rate of change of the mean action

$$\langle x\rangle = \int_0^1 x\rho(x)dx = \int_0^1 [1 - D(x,t)]\,dx,$$

is proportional to the mean payoff gradient, viz.

$$\frac{d\langle x\rangle}{dt} = \int_0^1 \phi_x(x,D)dD(x,t). \tag{10}$$

The relation (10) equates global averages. To obtain a full dynamic picture we apply gradient dynamics to the averages over subintervals, so that

$$\frac{\partial}{\partial t}\int_0^x [1 - D(y,t)]\,dy = \int_0^x \phi_y(y,D)dD(y,t). \tag{11}$$

Differentiating (11) with respect to x, we have the local mass conservation law

$$D_t(x,t) = -\phi_x(x,D)D_x(x,t), \tag{12}$$

which holds in the absence of discontinuities in the distribution.[12] In the discussion to follow we refer to (12) as the Master Equation. The Master Equation states that $D_t(x,t)$, the rate of increase of total mass in the interval $[0,x]$, is equal to the rate at which mass moves leftward past point x. In turn, the mass flow rate equals the product of the density ($D_x = \rho$) and the gradient ϕ_x. Thus, we make the key assumption that the mass velocity equals the gradient ϕ_x. The steeper the slope of the payoff function, the more rapidly consumers adjust their actions as they climb the local payoff landscape.[13]

To insure that all population mass remains in the interval $[0,1]$, we impose boundary sign restrictions. At the upper boundary of the action set, $x = 1$, we impose the condition $\phi_x(1,t) \le 0$ in (12). At the lower boundary, $x = 0$, we impose the condition $\phi_x(0,t) \ge 0$.

3.1. Master deterministic dynamical equations.

Using the Emulation gradient $\phi_x^{\mathsf{E}}(x,D) = c/x - D(x)$ of (5) in the Master Equation (12), we obtain the partial differential equation

$$D_t = D_x[D - (c/x)]. \tag{13}$$

Using the Excess gradient $\phi_x^X(x,D) = c/x - 1 + D(x)$ in (12), we obtain

$$D_t = D_x[1 - D - (c/x)]. \tag{14}$$

[12] The conventional expression of mass conservation is

$$\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0,$$

where $q(x,t)$ is the net flux of mass per unit time, in or out of the infinitesimal action interval $[x, x+dx]$, and it is assumed that ρ, q are continuously differentiable. Given an appropriate choice of flux $q(x,t)$, this conservation equation describes continuous time evolution in the absence of shocks. In the text we choose a mass flux of the form $q(x,t) = \phi_{\mathsf{x}}(x,D)\rho(x,t)$.

[13] One can show that gradient dynamics emerge naturally when players respond optimally to current circumstances and face adjustment costs that increase quadratically in the adjustment speed. In particular, myopically rational players facing quadratic adjustment costs choose adjustment rates $A\phi_{\mathsf{x}}(x,t)$, where A is a positive factor proportional to Δt for $x \in (0,1)$, and appropriate boundary conditions hold at the endpoints $x = 0$, 1.

Equations (13) and (14) specify the deterministic form of the dynamics in action space under the two alternative reward patterns, given an initial cumulative distribution function $D(x, 0) = F(x)$, for fixed $c \in [0, 1)$.[14]

4. Steady state and dynamical solutions

4.1. **Asymptotic Steady States.** For any given distribution D and $0 < c < 1$, the Emulation gradient $\phi_x^{E} = c/x - D(x, t)$ is monot t .D i.e., $D^*(x) = 1 - c/x$. Thus for $0 < c < 1$, the asymptotic distribution $D^*(x)$ follows the hyperbolic arc $1 - c/x$ inside the unit square $0 \le x, D^* \le 1$. The asymptotic distribution follows the lower edge $D^*(x) = 0$ (upper edge $D^*(x) = 1$) when the arc is below (above)[16] the square. Thus all consumers end up on the same indifference curve given by the hyperbolic arc $D^*(x) = 1 - c/x$. On this indifference curve the marginal gain $1 - D^*(x)$ from increasing rank dependent consumption is exactly offset by the corresponding marginal cost $-c/x$ of ordinary consumption. The payoff function is constant and maximal along the arc and at the upper endpoint $x = 1$, but takes lower values $c \log x - x$ in the unpopulated zone $[0, c)$.

4.2. **General solutions for Emulation and Excess.** The non-linear Emulation equation (13) can be regarded as an integrable family of ordinary differential equations linked by the initial condition $D(x, 0) = F(x)$. To characterize solutions, it is useful to define an auxiliary variable $z = z(x, t)$ implicitly given by $D(x, t) = F(z)$. In the Appendix we show that (13) can be integrated to obtain the implicit solution

$$t = \frac{z - x}{F(z)} + \frac{c}{F^2(z)} \ln\left(\frac{|c - zF(z)|}{|c - xF(z)|}\right). \tag{15}$$

The solution (15) is general in the following sense. Assume there exists a function $z = z^*(x, t)$ that satisfies (15) for each $x \in [0, 1]$ and $t \ge 0$, given an arbitrary initial distribution $F(x) = D(x, 0)$ and fixed $c \in [0, 1)$. If $z^*(x, t)$ is single valued, (13) has the solution $D(x, t) = F(z)$. On the other hand, if $z^*(x, t)$ is multiple valued for some (x, t) then the solution incorporates a shock wave (see Section 5).

To fix ideas, we find the explicit solution for the case $c = 0$ and the uniform initial distribution $F(x) = 0$, $x < 0$; $F(x) = x$, $x \in [0, 1]$. Given a uniform initial distribution $D(x, 0) = x$, (15) gives $z = \frac{x}{1-t}$. Hence the required $c = 0$ solution is

$$D(x, t) = \frac{x}{1 - t}\Theta(1 - t - x), \;\; 0 < t < 1, \tag{16}$$

with associated density

$$\rho(x, t) = \frac{1}{1 - t}\Theta(1 - t - x), \;\; 0 < t < 1. \tag{17}$$

Figure 1 Caption. Evolution of probability density for Emulation rule with initial uniform distribution and pure rank dependent consumption.

Equation (16) tells us that during the time interval $0 < t < 1$ households decrease their consumption of ordinary goods and increase rank dependent consumption so as to maintain a uniform distribution in x on the (shrinking) interval $0 < x \le 1 - t$. Density functions for increasing values of t are shown in Figure 1. At $t = 1$ the time paths for all households cross, and all consumers cluster in a delta-function singularity at $x = 0$. For $t > 1$, conservation of probability mass in the unit interval and gradient adjustment imply there is no further change in the distribution. For $t \ge 1$, every household devotes all resources to rank dependent consumption, $D(x, t) = \Theta(x)$, and all population is clustered in an atom at $x = 0$.

Figure 2 illustrates the evolution of the cumulative distribution function when ordinary consumption has positive weight $c = .05$. From Theorem 1, the long run equilibrium distribution becomes concentrated at an interior point, rather than at $x = 0$ as in the $c = 0$ case.

[16] The arc cuts the upper edge of the square only in the less interesting case $c > 1$.

Figure 2 Caption. Time slices of the cumulative distribution function $D(x, t)$ for $c = .05$, given Emulation dynamics.

Figure 3 shows the evolution of the distribution function for Excess dynamics, given a suitable initial distribution with $c = .05$. The distribution converges smoothly to the predicted hyperbolic distribution. Note that the extreme $x = 1$ allocation (i.e., no rank dependent consumption) is chosen by the fraction $1-(1-c/x) = c$, or 5%, of the population. The remainder of the population make dispersed choices ranging up to 95% rank dependent consumption.

Figure 3 Caption. Time slices of the of the cumulative distribution function $D(x, t)$ under Excess dynamics, for utility parameter $c = 0.05$. Note the asymptotic equilibrium on the hyperbolic curve $D^* = 1 - c/x$.

5. Population shock waves under Emulation

Figures 2 and 3 illustrate smooth progression to long run equilibrium. Another dynamical possibility is the emergence of sharply defined subgroups at finite time. These sharply defined subgroups are associated with moving jump discontinuities, or shock waves, in the action distribution. Shock waves are an artifact of the deterministic form of the model. They appear when the initial distribution is sufficiently steep that the nonlinearity in (13) creates a "breaking wave." As in ocean surf, the break occurs because the velocity of the upper part of $D(x, t)$ exceeds the velocity of the lower part, and the waveform becomes vertical. After the break there is a moving discontinuity in the distribution. The size of the discontinuity represents the cluster mass, the fraction of consumers with an identical consumption pattern. Keeping in mind a larger context, to be discussed in Section 6, in which there is Brownian uncertainty about the shape of the local payoff landscape, one may interpret shocks as markers for clustering in the limit in which uncertainty about the local topography goes to zero.

Moving shocks interior to the action interval $[0, 1]$ occur under Emulation for quite general initial conditions. A shock occurs at a time and location such that the waveform $D(x, t) = F(z)$ becomes vertical. A vertical wavefront requires that the partial derivatives z_t, z_x become singular. From (15) we have

$$z_t = \frac{F(z)}{1 - tf(z) + c\gamma_z(z, x)}, \tag{18}$$

where $f(z) = F'(z)$ is the initial density, and

$$\gamma(z, x) \equiv \frac{1}{F(z)} \ln \left(\frac{|c - zF(z)|}{|c - xF(z)|} \right).$$

From (18), a shock occurs at time t^* and action x^* if

$$t^* f(z) = 1 + c\gamma_z(z, x^*). \tag{19}$$

A shock first appears for the smallest value of t^* such that some real $z \in [0, 1]$ satisfies equation (19). The theorem that follows formally characterizes the conditions for a shock wave for small c.

Theorem 2. Shocks. Let the initial distribution $F(x)$ be thrice continuously differentiable, with a regular strict maximum at $x = q \in (0, 1)$. Then for all sufficiently small $c > 0$, the solution of (15) has a moving interior shock. Up to first order in c, the shock emerges at time

$$t^*(c) = 1/f(q) + cT(q) + O(c^2)$$

and location

$$x^*(c) = q - F(q)/f(q) + cX(q) + O(c^2) \in (0,1).$$

The functions $T(q),\ X(q)$ are given in the Appendix.

Proof. See the Appendix.

To illustrate the qualitative features of shocks, we describe the dynamics in the special case where $c = 0$ and initial density $f(x) = 6x(1-x)$. In this special case, the mode of the initial density occurs at $x = 1/2$, and symmetry about $x = 1/2$ implies $F(1/2) = 1/2$. At the mode the initial density $f(1/2) = 3/2$. Using these values in Theorem 2, we observe that the shock begins at time $t^*(0) = 2/3$ and at location $x^*(0) = 1/6$.

For given $t \geq t^*$, the state is described by three simultaneous equations. The first two equations come from the smooth solution (15) applied to the leading and trailing edges of the shock, with locations z_L, z_R expressed in terms of the auxiliary variable z. The third equation follows from the principle of population mass conservation: the vertical line at $x = s$ in the (x, z) plane cuts the S-shaped level curve described by (15) so that the two lobes have equal area. (See the Appendix.)

In the present special case we obtain shock position

$$s(t) = \frac{1}{2}(1-t) \tag{20}$$

and magnitude

$$F(z_R) - F(z_L) = \frac{1}{t}\sqrt{3 - \frac{2}{t}} \tag{21}$$

for $t \in (2/3, 1)$. For $t \geq 1$, all mass is clustered at the boundary point $x = s = 0$. Figure 4 shows the time development of the shock. Note that the shock magnitude grows until it exhausts the total probability mass at $t = 1$, as indicated by (21).

Figure 4 Caption. Time slices for the cumulative distribution function $D(x,t)$ under Emulation, for pure rank dependent consumption $c = 0$, showing the development of a shock wave. Note that the shock initiates at $t = t^* = 2/3$, and that for $t \geq 1$ the shock magnitude exhausts the entire probability mass.

A shock wave in this context may be interpreted as follows. Consider a consumer, labelled P, whose initial action is at $x = 1/2$, the peak of the density. Until a shock occurs, P remains alone on the characteristic curve indexed by $z = 1/2$. However, the nonlinear dynamics imply that P chooses to decrease x (and increase rank dependent consumption) more rapidly than do consumers with initially lower x. Consequently, at time t^* and ordinary consumption level x^*, P begins to overtake those other consumers. Given our assumption of identical underlying preferences, P cannot "pass" other individuals because his or her behavior is identical to theirs once he attains the same lower level of ordinary consumption x. Instead, he clumps together with them, and a shock is created. The process continues. The new subgroup subsequently overtakes consumers with lower x, and is in turn overtaken by consumers whose initial x is slightly higher. Thus, beginning at (x^*, t^*) a growing cluster of consumers with identical consumption patterns develops, and there is collective movement towards lower ordinary consumption and higher rank dependent consumption.

Compressive shocks of this sort do not occur under Excess dynamics. Instead, from any initial state [17] the distribution function $D(x,t)$ converges smoothly to the hyperbolic arc $D^*(x) = 1 - c/x$ truncated at the boundaries of the unit square. Suppose that at time $t \geq 0$ the distribution $D(x,t)$ crosses the arc $1 - c/x$ at some point $\widetilde{x} \in (c, 1)$. At $x = \widetilde{x}$ the marginal gain $1 - D(x,t)$ from increasing rank dependent consumption is exactly offset by the decreasing marginal cost c/x. If the slope of $D_X(\widetilde{x}, t)$ is less than the slope $c/\widetilde{x}^2$ of the hyperbolic arc at $x = \widetilde{x}$, the marginal gain $1 - D(x,t)$ of slightly more rank dependent consumption is less than the marginal cost of ordinary consumption, and utility therefore increases as rank dependent consumption decreases. The reverse is true for consumers with slightly less rank dependent consumption. Therefore, just as in Emulation dynamics, population mass moves towards $\widetilde{x}$ from both directions, and the slope of D at $\widetilde{x}$ increases as t increases, while the intersection point $x = \widetilde{x}$ remains fixed. However, if $D_x(\widetilde{x}, t)$ exceeds the slope of $1 - c/x$ at $\widetilde{x}$, the marginal gain $1 - D(x,t)$ of slightly more rank dependent consumption exceeds the marginal cost c/x, leading to a further increase of rank dependent consumption.[18] Likewise, consumers with $1 - x$ slightly less than $1 - \widetilde{x}$ further decrease rank dependent consumption. So mass moves away from $\widetilde{x}$ in both directions, and $D_x(\widetilde{x}, t)$ decreases as t increases. The population distribution $D(x,t)$ thus decreases where it is above the arc $1 - c/x$ and increases where it is below that arc. In all cases $D(x,t)$ ultimately converges to $D^*(x) = 1 - c/x$ where, as observed earlier, every individual attains the same maximal level of utility.

6. Two-group asymptotic equilibrium under uncertainty

Outcomes are more complex if we allow Excess and Emulation subgroups to interact in the presence of random imprecision in agents' knowledge of the local landscape. We shall discuss the asymptotic equilibria that result when the two subgroups have fixed proportions $P_E + P_X = 1$. For simplicity, we assume the tradeoff parameter c, the diffusion rate k, and the (implicit) timescale that sets the adjustment rate are the same for the two groups. We incorporate Brownian noise in the dynamic equations in the form of an additive diffusion term kD_{xx}, where the diffusion rate (or rate of information loss) is k .

[17] Even with an initially homogeneous distribution such as $F(x) = \Theta(x - 1/2)$, in Excess mode a rarefaction wave forms in which the distribution spreads out and disperses and converges to the same hyperbolic arc. See, e.g., Logan (1994, pp. 86-87) for a discussion of rarefaction.

[18] This behavior cannot occur under Emulation because there the marginal return to rank dependent consumption is given by the increasing function $D(x,t)$, rather than the decreasing function $1 - D(x,t)$.

The coupled dynamics then follow from the system[19]

$$D^E_t = kD^E_{xx} + D^E_x[D - (c/x)];\ D^X_t = kD^X_{xx} + D^X_x[S - (c/x)], \tag{22}$$

and equilibrium occurs for

$$kD^E_{xx} = D^E_x\left(\frac{c}{x} - D\right);\ kD^X_{xx} = D^X_x\left(\frac{c}{x} - 1 + D\right), \tag{23}$$

for Emulation and Excess respectively. In (23) the respective mass fractions of the two subgroups are $D^{E,X}(1) = P_{E,X}$, and we have written $D = D^E + D^X$.

In terms of densities, (23) becomes, with $\rho = \rho^E + \rho^X$,

$$\frac{k}{\rho^E}\frac{d\rho^E}{dx} = \frac{c}{x} - \int_0^x \rho(x')dx'; \tag{24}$$

$$\frac{k}{\rho^X}\frac{d\rho^X}{dx} = \frac{c}{x} - 1 + \int_0^x \rho(x')dx'. \tag{25}$$

From (25, 24) we observe that equilibrium occurs when the fractional increase of agents populating the action interval $[x,\, x+dx]$, due to diffusion, equals the payoff per capita lost as the result of diffusion. In integral form, defining $\alpha \equiv c/k$,

$$\rho^E(x) = x^\alpha e^{-\frac{1}{k}\int_0^x D(x^0)dx^0}; \tag{26}$$

$$\rho^X(x) = x^\alpha e^{-\frac{1}{k}\int_0^x S(x^0)dx^0}. \tag{27}$$

Equations (26,27) suggest a qualitative picture in which the small x behavior of the densities is set by an intragroup balance between diffusion and ordinary consumption that results in the power law $\rho \sim x^\alpha$, while the large x behavior is set by a global intergroup interaction with major contributions from the x regions in which the two densities take their maximum values. The sign of the right-hand side of (24) is determined by $c - xD(x)$, and the monotonicity of $xD(x)$ therefore implies that the Emulation equilibrium density is unimodal for all c, with an interior mode. In contrast, the sign of the right-hand side of (25) is determined by $c - xS(x)$. But $xS(x) = 0$ for $x = 0, 1$ and therefore necessarily takes a maximum value $\mu < 1$ at an action interior to $[0, 1]$. Thus the Excess density is monotone increasing for $c > \mu$. For $c \leq \mu$, (25) shows ρ^X increases at both boundaries with elasticity $c/k \equiv \alpha$,

[19]With $c = 0$ equations (22) take the form of the viscous Burger's Equation, an analytically solvable nonlinear partial differential equation much studied in fluid dynamics. See, e.g. Whitham, 1974, chapt. 4. It is unlikely that analytic solutions for $c \neq 0$ can also be obtained. The full equations (22) do not satisfy the Painleve property (Weiss et al, 1983), even in the uncoupled form. Furthermore, one proves, using the L_8 Lie alebraic test (Ibragimov 1994), that there exists a linearizing change of variables for the uncoupled form of the equilibrium equations (23) only for $c = 0$.

With $c \neq 0$, but in the absence of the nonlinear terms DD_x, equations (22) can be transformed into the imaginary time Schrodinger equation for the hydrogen atom, which is integrable in terms of Whittaker (confluent hypergeometric) functions. See, e.g., Grosche and Steiner, 1998. This suggests a useful interpretation of the dynamics of the two-group model. There is a background dynamics in which there are random flights between actions x, x^0 chosen at respective times t, t^0, with transition probabilities $G(x, t; x^0, t^0)$ set by the Green's function of the $c = 0$ limit of the model. These flights begin and end with "transactions" among multiple pairs of actors represented by the nonlinear terms DD_x. Following this reasoning, one can formulate the system (22) as a pair of coupled nonlinear integral equations that in principle can be solved by iteration.

implying ρ^X has, at minimum, one interior maximum and one boundary maximum at $x = 1$, a fortiori bracketing at least one interior minimum. This configuration corresponds to the sign pattern for Excess in the deterministic case. Recalling the discussion in Section 4, we see that the introduction of uncertainty in the shape of the landscape causes the asymptotic cluster under Emulation to acquire a finite spread of magnitude roughly $\sqrt{k}$. Given Excess dynamics, for sufficiently large c, the payoff cost of clumping is sufficiently large that the equilibrium density shows the same pattern of monotone increase as in the deterministic case. When c is small, diffusion allows clustering at the boundary $x = 1$ and at an interior point, for sufficiently small x. The Excess density also shows a diffusive spread of order $\sqrt{k}$, as expected.

In Figure 5 we show the two-group equilibrium for roughly equal weights of the two groups. We choose parameters $c = .05$, $k = .01$. Note the long tail of the Excess density and the subgroup of population with pure ordinary consumption at $x = 1$. The maxima occur for zero payoff (cf.(23)). In the Excess density the total distribution $D << 1$ for x, c sufficiently small (cf.(27)). The corresponding maximum thus sits at $x \cong c$. The Emulation maximum occurs to the right of the bulk of the Excess mass, implying a position $x \cong 2c$.

Figure 5 Caption. Probability densities for two-group model with relative weights .47, .53 for Emulation and Excess respectively, and with tradeo! parameter $c = .05$, and information di!usion rate $k = .01$

density has a large rank dependent cluster, an extended tail and a small subgroup of individuals who prefer ordinary consumption.

The model analyzed in this paper is one example of an infinite class characterized by the following features. (1) The equivalent dynamics in discrete time follow a Markov process, so that the continuous time picture is given by an evolution equation. (2) The dynamics conserve population mass. (3) The dynamics incorporate a random walk in action space associated with uncertainties in the perceived local preference landscape. (4) There are nonlinear links between individual agents due to pairwise ranking. From preliminary analysis, we believe that constructing empirically testable versions of such models will require considering the joint distribution of income together with the allocation between ordinary and rank dependent consumption. We conjecture that the full solutions of models of the extended type, including macroshocks, will incorporate the spontaneous creation and disappearance of clumps of agents, and that the clumping process will then fully describe the microeconomic dynamics. This is consistent with previous work on Burger's equation with stochastic forcing (e.g., Fogedby 1998).

Given a model that describes the creation and dissipation of clusters, it should be possible to apply these ideas to asset pricing, and in particular to asset price bubbles. The equilibrium structure of the Excess density, described in Section 6, is achieved via a balance between the tendency to cluster in preference for rank dependent consumption and the tendency to disperse with a preference for ordinary consumption. This suggests that when there is non-price linkage and Brownian uncertainty, a mechanical rule for clustering can implicitly carry with it a rule for dispersive behavior. Thus there emerges a possible alternative for describing asset price bubbles, distinct from recruitment and contagion schemes, and also from models based on rational expectations.[20] In asset markets non-price linkage can occur when investors adjust their asset portfolios to achieve or exceed recent returns received by other investors. This seems a useful key for describing the collective actions of money managers, whose compensation is heavily rank dependent. To apply models of the type considered here, one may characterize the investor population in terms of portfolio risk measures (e.g., beta), and model how the asset distribution changes over time, assuming a return to mimetic behavior. Such an extension will be more complex, because it necessarily involves both price (or asset return) and non-price interdependence. If the model is applicable, we expect clustering in both the bubble creation and dissipation phases.

8. Appendix: Proofs and Derivations

This appendix collects derivations and proofs omitted in the text, in the same order as the related text material.

8.1. **Theorem 1. Clustering.** Let the distribution $D(x,t)$ be a solution to (13) for a given initial condition $D(x,0) = F(x)$. Then $D(x,t)$ converges pointwise as $t \to \infty$ to $\Theta(x - \widetilde{x})$, where $\widetilde{x} = \sup\{x \in [0,1] : xF(x) < c\}$.[21]

[20] See generally, Shiller, 2000, for a recent discussion of alternative approaches to asset price bubbles.

[21] Remark. The proof below is written out for the case that $D(x,t)$ is continuous in x for all $t \geq 0$ with density $\rho(x,t)$. The logic also applies to the class of distributions $D(x,t)$ with jump discontinuities, but the proof requires heavier notation.

Notation: $sgn(z) = -1, 0,$or 1 indicates whether z is negative, zero or positive.

Lemma. For all $x \in [0,1]$ and all $t \geq 0$, we have $sgn\,[D(x,t) - c/x] = sgn\,[x - \widetilde{x}]$.

Proof of Lemma. Set $0 < c < 1$. Let $x = \widetilde{x} \in (0,1)$ solve $F(x) = c/x$. The solution $x = \widetilde{x}$ is unique and interior because the function $F(x) - c/x$ is continuous, strictly increasing, negative at $x = 0$ and positive at $x = 1$. Hence the above sign relation holds at time $t = 0$. Furthermore, no mass moves past the point $x = \widetilde{x}$ at $t = 0$ because the velocity field at $x = \widetilde{x}$ is $D(\widetilde{x}, 0) - c/\widetilde{x} = 0$. The same flow restriction holds at later times by the same argument. Thus $D(\widetilde{x}, t) = F(\widetilde{x})$ holds in $t \geq 0$, and the conclusion holds for all $t \geq 0$.¥

Remark. In the case in which $D(x,t)$ is discontinuous in x one concludes a fortiori that no mass moves past $\widetilde{x}$. By examining the left and right limits of the gradient at $x = \widetilde{x}$, one sees that the point $x = \widetilde{x}$ absorbs mass from both directions.

Proof of Theorem. It suffices to construct a Ljapunov function V and verify: (a) as a functional on the space of cdf's on $[0,1]$, V attains a global minimum at D^*, the unit step function at $\widetilde{x}$; and (b) as a function of time along a solution $D(x,t)$, the value of V is strictly decreasing at all times when $D(x,t) \neq D^*$. To verify (a), let $V(t) = \int_0^1 (x - \widetilde{x})^2 \rho(x,t)dx$. The integrand is zero at $x = \widetilde{x}$ and where $\rho(x,t) = 0$. Elsewhere it is positive. Thus (a) is verified. To verify (b) integrate by parts to obtain $V(t) = (1 - \widetilde{x})^2 - 2\int_0^1 (x - \widetilde{x})D(x,t)dx$. Hence the time derivative exists and is equal to $\dot{V} = -2\int_0^1 (x - \widetilde{x})D_t(x,t)dx$. Using (13), we have $\dot{V} = -2\int_0^1 (x - \widetilde{x})[D(x,t) - c/x]\rho(x,t)dx$. The Lemma now tells us that $\dot{V}$ is negative except at $D(x,t) = D^*$, and (b) follows. ¥

Remark. In the discontinuous case one needs to use Stieltjes integrals but the argument is unchanged.

8.2. Derivation of General Solution to (13). Write (13) in the form

$$D_t - [D - (c/x)]D_x = 0. \tag{28}$$

A solution $D(x,t)$ of (28) defines a surface in $[D(x,t), x, t]$ space. Along any such integral surface the total time derivative is

$$dD(x,t)/dt = D_t + D_x dx/dt. \tag{29}$$

Comparing (28) and (29), we see that (28) defines a set of curves along each of which

$$dD/dt = 0, \tag{30}$$

$$dx/dt = c/x - D. \tag{31}$$

Equation (31) describes the particular time path, or characteristic curve, $x(t)$ followed by a consumer with position $x(0)$ at time $t = 0$. Equation (30) tells us that along such a characteristic curve, $D(x,t) =$ constant.

To characterize solutions of (30, 31), it is useful to define an auxiliary variable $z = z(x,t)$ implicitly given by $D(x,t) = F(z)$. Because D is constant along each characteristic curve defined by (30, 31), we label each such curve by the corresponding value of z. Separating variables and substituting $F(z)$ for D, we have $dt = xdx/(c - xF(z))$. Because z is fixed along any characteristic, this expression can be integrated directly using the textbook formula

$$\frac{x}{1 - ax} = \frac{1}{a^2}\frac{d}{dx}[1 - ax - \ln|1 - ax|].$$

We obtain

$$t + t_0(z) = \frac{1 - xF(z)/c - \ln|1 - xF(z)/c|}{F^2(z)/c}. \tag{32}$$

In (32), the integration constant $t_0(z)$ is constant along each characteristic but varies across characteristics. By the definition of z, $x = z$ at $t = 0$. Hence

$$t_0(z) = \frac{1 - zF(z)/c - \ln|1 - zF(z)/c|}{F^2(z)/c}, \tag{33}$$

and we may subtract (33) from (32) to obtain the implicit solution

$$t = \frac{z - x}{F(z)} + \frac{c}{F^2(z)} \ln\left(\frac{|c - zF(z)|}{|c - xF(z)|}\right)$$

of the master equation (13), as given by equation (15) of the text.

8.3. **Shock dynamics.** We derive the behavior over time of the shock position and magnitude, given an initial probability density $f(x)$ and corresponding cumulative distribution $F(x)$. We assume Emulation mode and for closed form solutions we set the tradeoff parameter $c = 0$. The $c = 0$, pure rank dependent consumption limit enables an analytic solution while preserving the qualitative behavior of the shock wave.

In general, three conditions determine the shock position $s(t)$ and the leading and trailing values z_L,z_R that mark the left and right edges of the shock in terms of the auxiliary variable z.[22] The first two conditions apply the solution (15) of the underlying dynamic equation to the leading and trailing edges of the shock, viz.

$$t = \frac{z_L - s}{F(z_L)} + \frac{c}{F^2(z_L)} \ln\left(\frac{|c - z_L F(z_L)|}{|c - sF(z_L)|}\right); \tag{34}$$

$$t = \frac{z_R - s}{F(z_R)} + \frac{c}{F^2(z_R)} \ln\left(\frac{|c - z_R F(z_R)|}{|c - sF(z_R)|}\right), \tag{35}$$

The third condition comes from applying the integral form of the underlying conservation law across the shock. Substitution of $\phi_y(y, D) = \frac{c}{y} - D(y, t)$ into equation (11) yields

$$-\frac{\partial}{\partial t}\int_{x_1}^{x_2} D(y,t)dy = -\int_{D(x_1,t)}^{D(x_2,t)} \left[\frac{c}{y} - D(y,t)\right] dD(y,t). \tag{36}$$

Let $D(x, t)$ have a jump discontinuity at $x = s(t)$, and choose $x_1 < s(t) < x_2$. Then in the limit as $x_1 \to s(t)$ from below and $x_2 \to s(t)$ from above, (36) becomes

$$-\left[F(z_R) - F(z_L)\right]\frac{ds}{dt} = -\frac{c\left[F(z_R) - F(z_L)\right]}{s} + \frac{1}{2}\left[F^2(z_R) - F^2(z_L)\right].$$

On division by the shock magnitude $F(z_R) - F(z_L)$, we obtain the shock velocity

[22] These are the Rankine-Hugoniot conditions. The procedure used here to derive the equal area condition (40) below is laid out in Whitham (1974). See Smoller (1994) for formal discussion. To our knowledge, the derivation of the shock initiation time and magnitude, given an initial symmetric beta density, is new.

$$\frac{ds}{dt} = -\frac{1}{2}[F(z_L) + F(z_R)] + \frac{c}{s}. \tag{37}$$

For $c = 0$, (34) and (35) can be combined and expressed in the symmetric forms

$$s = \frac{1}{2}\left[z_L + z_R\right] - \frac{t}{2}[F(z_L) + F(z_R)], \tag{38}$$

$$t = \frac{z_R - z_L}{F(z_R) - F(z_L)}. \tag{39}$$

Furthermore, on differentiating (38) with respect to t, equating the result with the $c = 0$ form of (37), and substituting for t from (39), we obtain

$$[z_R - z_L]\left[F'(z_L)\dot{z}_L + F'(z_R)\dot{z}_R\right] = [F(z_R) - F(z_L)]\left[\dot{z}_L + \dot{z}_R\right],$$

which integrates to the "equal area condition"

$$\frac{1}{2}[F(z_L) + F(z_R)]\left[z_R - z_L\right] = \int_{z_L}^{z_R} F(z)dz. \tag{40}$$

Equation (40) states that the shock cuts off equal areas from the breaking wavefront, preserving population mass. This interpretation becomes transparent if we rewrite (40) in the form

$$\int_{z_L}^{\widehat{z}} \left\{\frac{1}{2}[F(z_R) - F(z_L)] - F(z) + F(z_L)\right\} dz = \int_{\widehat{z}}^{z_R} \left\{F(z) - F(z_L) - \frac{1}{2}[F(z_L) - F(z_R)]\right\} dz,$$

which explicitly equates the areas swept out by the right and left "lobes" of the shock. Note that the crossover point of the shock, $z = \widehat{z}$, is determined by $F(\widehat{z}) = \frac{1}{2}[F(z_L) + F(z_R)]$.

To determine the shock initiation time t^*, it is helpful to write (39) in terms of the density $f(z)$. Introduce $\Delta = \left[z_R - z_L\right]/2$, $\bar{z} = \left[z_R + z_L\right]/2$, so that $\Delta(t^*) = 0$ when a shock initiates. Then (39) becomes

$$\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} f(z + \bar{z})dz = \frac{1}{t}, \tag{41}$$

and in the limit $\Delta \to 0$ we have

$$\frac{1}{t^*} = f(\bar{z}). \tag{42}$$

Certain basic relations follow from the assumed unimodality and symmetry about $z = 1/2$ of the density $f(z)$. Symmetry $f(z + 1/2) = f(-z + 1/2)$ gives $\bar{z} = 1/2$,

$$\int_0^{1/2-\Delta} f(z)dz = \int_{1/2+\Delta}^{1} f(z)dz,$$

and therefore

$$F(z_L) + F(z_R) = F(1/2 - \Delta) + F(1/2 + \Delta) = 1. \tag{43}$$

Using $\bar{z} = 1/2$, (38) and (43) give the position of shock at time t as

$$s(t) = \frac{1}{2}(1 - t),\ t > t^*, \tag{44}$$

where from (42), the shock initiation time satisfies

$$\frac{1}{t^*} = f(1/2). \tag{45}$$

In the symmetric $c = 0$ case, the shock therefore reaches position $s = x = 0$ at $t = 1$.

The shock initiation time and magnitude now follow for the special case described in the text. This solution is one instance from an infinite class of analytic, $c = 0$ shock solutions for the conservation law (13), for the family of initial unimodal beta densities symmetric about $z = 1/2$,

$$f_a(z) = \frac{z^{a-1}(1-z)^{a-1}}{B(a,a)},\ a > 1, \tag{46}$$

with corresponding distributions $F_a(z)$, where the beta function

$$B(a,a) = \frac{[(a-1)!]^2}{(2a-1)!}.$$

.

Equations (45,46) give the shock initiation time

$$t^* = \frac{2^{1-2a}}{B(a,a)}. \tag{47}$$

From (39), the shock magnitude is given implicitly by

$$2F_a(1/2 + \Delta) - 1 = \frac{2\Delta}{t}. \tag{48}$$

For the beta density $a = 2$ considered in the text, $F_2(z) = z^2(3 - 2z)$, and (48) reduces to a quadratic equation in Δ,

$$\frac{1}{t} = \frac{3}{2} - 2\Delta^2. \tag{49}$$

From (49) or (45), the shock initiates at $t^* = 2/3$. Equation (44) gives the initial shock position $x^* = 1/6$, as stated in the text. The shock magnitude thereafter is given by (21).[23]

8.4. **Theorem 2. Shocks.** Let the initial distribution $F(x)$ be thrice continuously differentiable, with a regular strict maximum at $x = q \in (0,1)$. Then for all sufficiently small $c > 0$, the solution of (15) has a moving interior shock. Up to first order in c, the shock emerges at time

$$t^*(c) = 1/f(q) + cT(q) + O(c^2)$$

and location

$$x^*(c) = q - F(q)/f(q) + cX(q) + O(c^2) \in (0,1).$$

Here,

$$T(q) = \left[\frac{\overline{\gamma}_z}{f(q)} - \frac{\overline{\gamma}_{zz}}{f''(q)} + \frac{\overline{\gamma}_z f'(q)}{f(q)f''(q)}\right];$$

$$X(q) = \left[\overline{\gamma} - \frac{\overline{\gamma}_z F(q)}{f(q)} + \frac{\overline{\gamma}\overline{\gamma}_{zz} F(q)}{f''(q)} - \frac{\overline{\gamma}_z f'(q) F(q)}{f(q)f''(q)}\right],$$

where, defining $Q = \ln \frac{qf(q)}{qf(q)-F(q)}$,

[23]A detailed derivation of (44) and (48) is available from the authors on request.

$$\overline{\gamma} = \frac{Q}{F(q)};$$

$$\overline{\gamma}_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)};$$

$$\overline{\gamma}_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{q^2 f'(q)Q + 2qf(q) + F(q)}{q^2 F^2(q)}.$$

Proof. Recall from equation (15) that the general implicit solution to the initial value problem is $0 = \alpha_1 \equiv tF(z) - z + x - c\gamma$, where

$$\gamma(c, z, x) \equiv \frac{1}{F(z)} \ln \frac{|zF(z) - c|}{|xF(z) - c|},$$

z is the auxiliary variable defined by $F(z) = D(x, t)$, and we have made the dependence on c explicit. Recall further that any boundary point (x, t) of a shock is a singularity, i.e., a point where the denominator of z_t (or z_x) as given in equation (18) is zero. That is, we have $0 = \alpha_2 \equiv 1 - tf(z) + c\gamma_z$, where $f = F'$ is the initial density. We seek the earliest time where a singularity occurs, i.e., a minimal positive value of t in the equation $0 = \alpha_2$. The associated first order condition is $0 = \alpha_3 \equiv cf(z)\gamma_{zz} - (c\gamma_z + 1)f'(z)$.

For fixed $c \geq 0$, let $\Phi(c, \cdot, \cdot, \cdot) : [0, 1]^2 \times \mathsf{R}_+ \to \mathsf{R}^3$ be the function that maps the point (z, x, t) to $(\alpha_1, \alpha_2, \alpha_3)$. Consider first the case $c = 0$. Here $0 = \alpha_3$ implies $f'(z) = 0$, so the shock first appears on the characteristic curve associated with the interior maximum $z = q$. From $0 = \alpha_2$ we infer that the shock initiates at time $t^*(0) = 1/f(q)$, and from $0 = \alpha_1$ we infer that the initial shock location is $x^*(0) = z - t^* F(z) = q - F(q)/f(q)$. It is clear that $0 < x^*(0) < q < 1$. We have $x^*(0) > 0$ because, at the global maximum $q > 0$ of $f(q)$, $F(q) = \int_0^q f(y)dy < qf(q)$. Thus we have an interior shock emerging in finite time for $c = 0$.

For small positive c, we apply the implicit function theorem to $\Phi(c, \cdot, \cdot, \cdot)$. The key condition (see e.g. Spivak, 1965) is that the Jacobian determinant $|\mathsf{J}(\alpha_1, \alpha_2, \alpha_3; z, x, t)|$ is not zero when evaluated at the point $c = 0$, $z = q$, $x = q - F(q)/f(q)$, and $t = 1/f(q)$. Explicitly,

$$|\mathsf{J}| = \begin{vmatrix} \frac{\partial\alpha_1}{\partial z} & \frac{\partial\alpha_1}{\partial x} & \frac{\partial\alpha_1}{\partial t} \\ \frac{\partial\alpha_2}{\partial z} & \frac{\partial\alpha_2}{\partial x} & \frac{\partial\alpha_2}{\partial t} \\ \frac{\partial\alpha_3}{\partial z} & \frac{\partial\alpha_3}{\partial x} & \frac{\partial\alpha_3}{\partial t} \end{vmatrix} = \begin{vmatrix} tf(z) - 1 & 1 & F(q) \\ tf'(q) & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & F(q) \\ 1 & 0 & -f(q) \\ f''(q) & 0 & 0 \end{vmatrix} = -f(q)f''(q).$$

The Jacobian is strictly positive because the density is positive at any maximum and has a negative second derivative at a regular maximum. Hence the desired implicit functions exist and have derivatives (evaluated at the same point) given by

$$\begin{pmatrix} z^{*\prime}(0) \\ x^{*\prime}(0) \\ t^{*\prime}(0) \end{pmatrix} = -\mathsf{J}^{-1} \begin{pmatrix} \frac{\partial\alpha_1}{\partial c} \\ \frac{\partial\alpha_2}{\partial c} \\ \frac{\partial\alpha_3}{\partial c} \end{pmatrix} = -\begin{pmatrix} 0 & 0 & \frac{1}{f''(q)} \\ 1 & \frac{F(q)}{f(q)} & \frac{-F(q)}{f(q)f''(q)} \\ 0 & \frac{-1}{f(q)} & \frac{1}{f(q)f''(q)} \end{pmatrix} \begin{pmatrix} -\gamma \\ \gamma_z \\ f(q)\gamma_{zz} - \gamma_z f'(q) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{f(q)\gamma_{zz} - \gamma_z f'(q)}{f''(q)} \\ \gamma - \frac{\gamma_z F(q)}{f(q)} + \frac{\gamma_{zz} F(q)}{f''(q)} - \frac{\gamma_z f'(q) F(q)}{f(q) f''(q)} \\ \frac{\gamma_z}{f(q)} - \frac{\gamma_{zz}}{f''(q)} + \frac{\gamma_z f'(q)}{f(q) f''(q)} \end{pmatrix}.$$

The values of γ and its derivatives are readily calculated at $c = 0$. Using the notation $Q = \ln|z| / |x| = \ln \frac{qf(q)}{qf(q)-F(q)}$, we find that at the relevant point $[z = q; x = q - F(q)/f(q); t = 1/f(q)]$ we have

$$\gamma = \frac{Q}{F(q)},$$

$$\gamma_z = \frac{1}{qF(q)} - \frac{Qf(q)}{F^2(q)},$$

and

$$\gamma_{zz} = \frac{2Qf(q)}{F^3(q)} - \frac{q^2 f'(q)Q + 2qf(q) + F(q)}{q^2 F^2(q)}.$$

The expressions in the Proposition now follow from the first order Taylor expansion at $c = 0$. They are valid as long as the shock position $x^*(c)$ remains above the zero $\widetilde{x}(c)$ of the gradient ϕ_x corresponding to the condition $x/c = F(x)$. Clearly $\widetilde{x}(0) = 0$, and $\widetilde{x}(c)$ is continuous in c because F has a density, so the condition $x^*(c) > \widetilde{x}(c)$ holds for sufficiently small c. ¥

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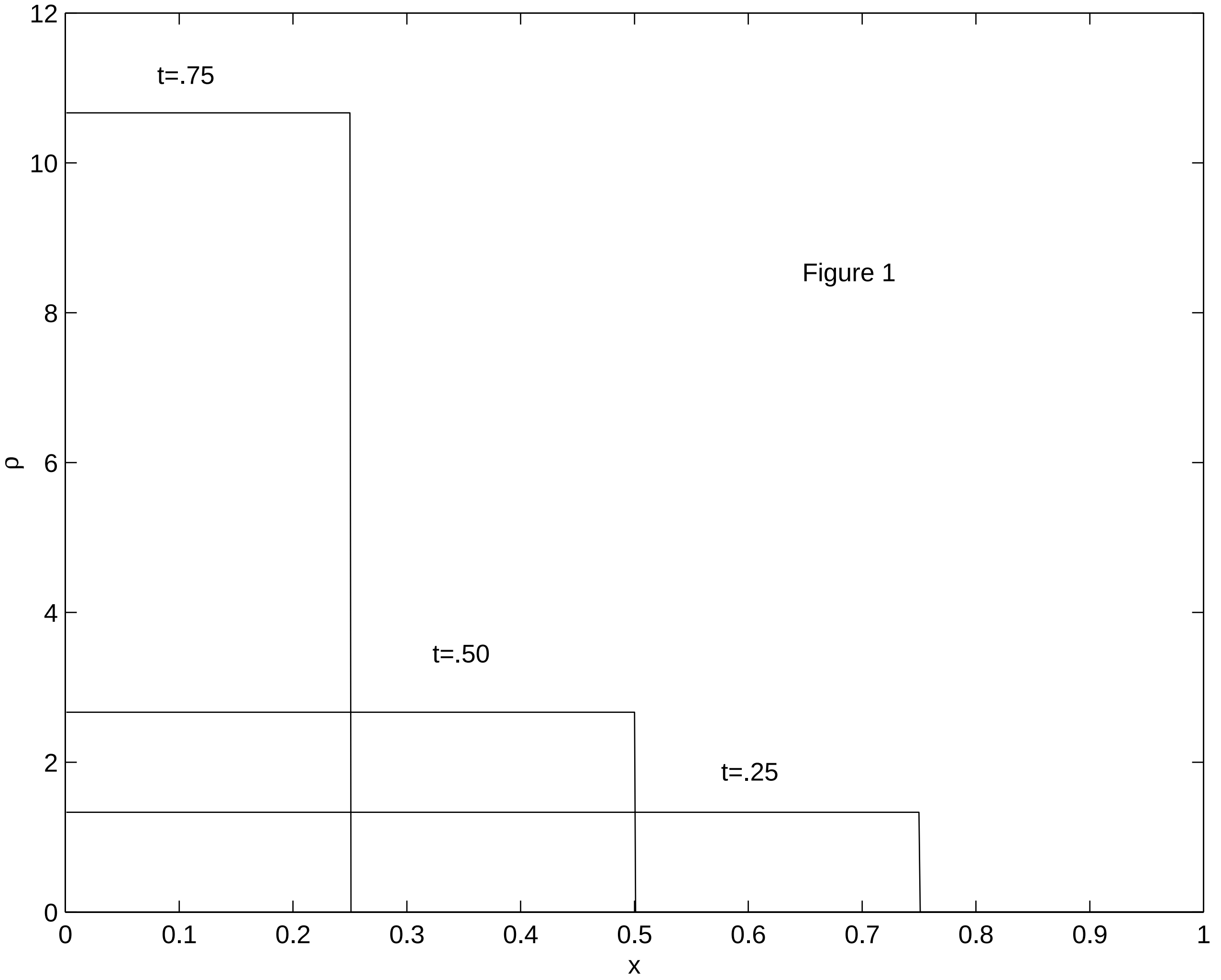
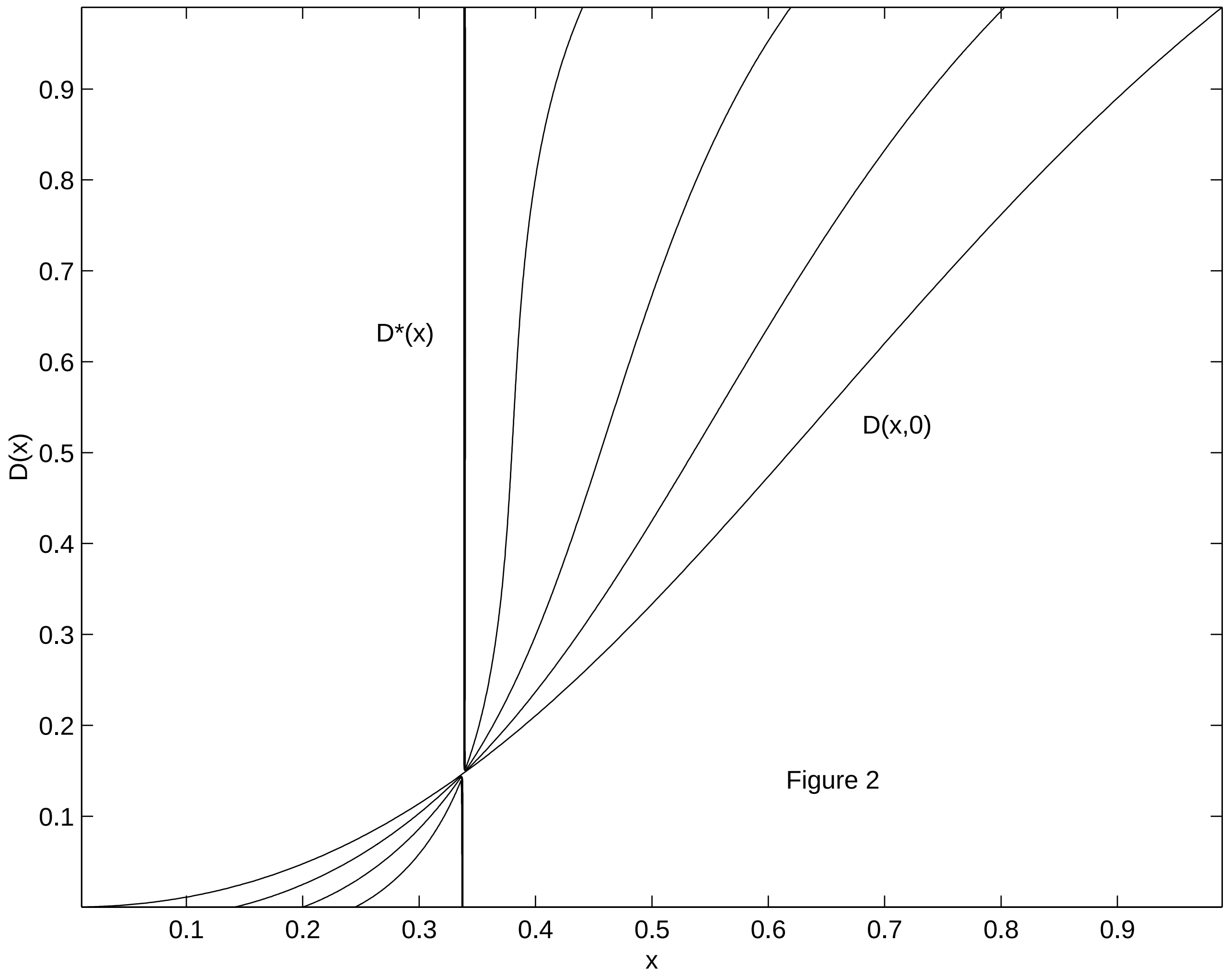
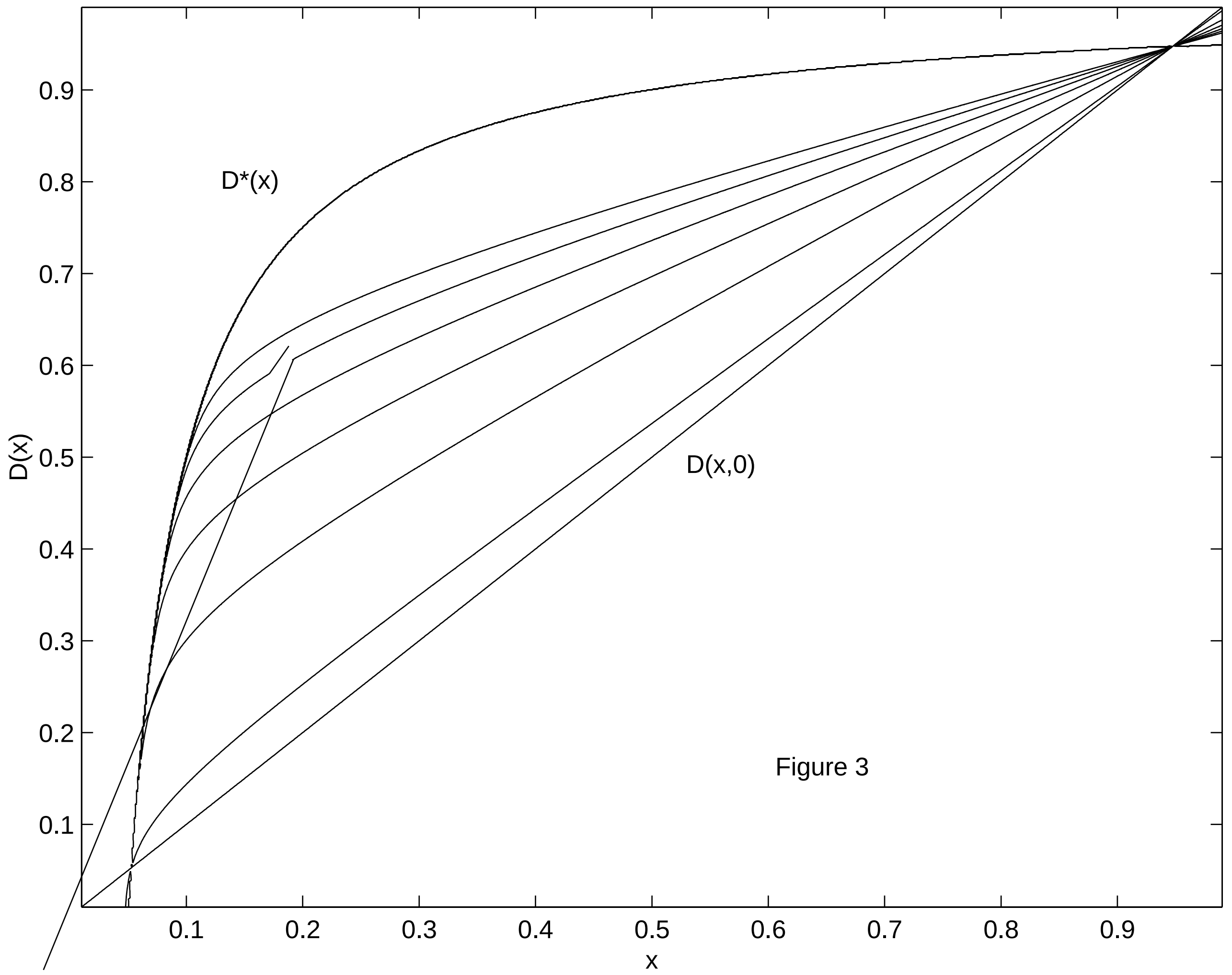
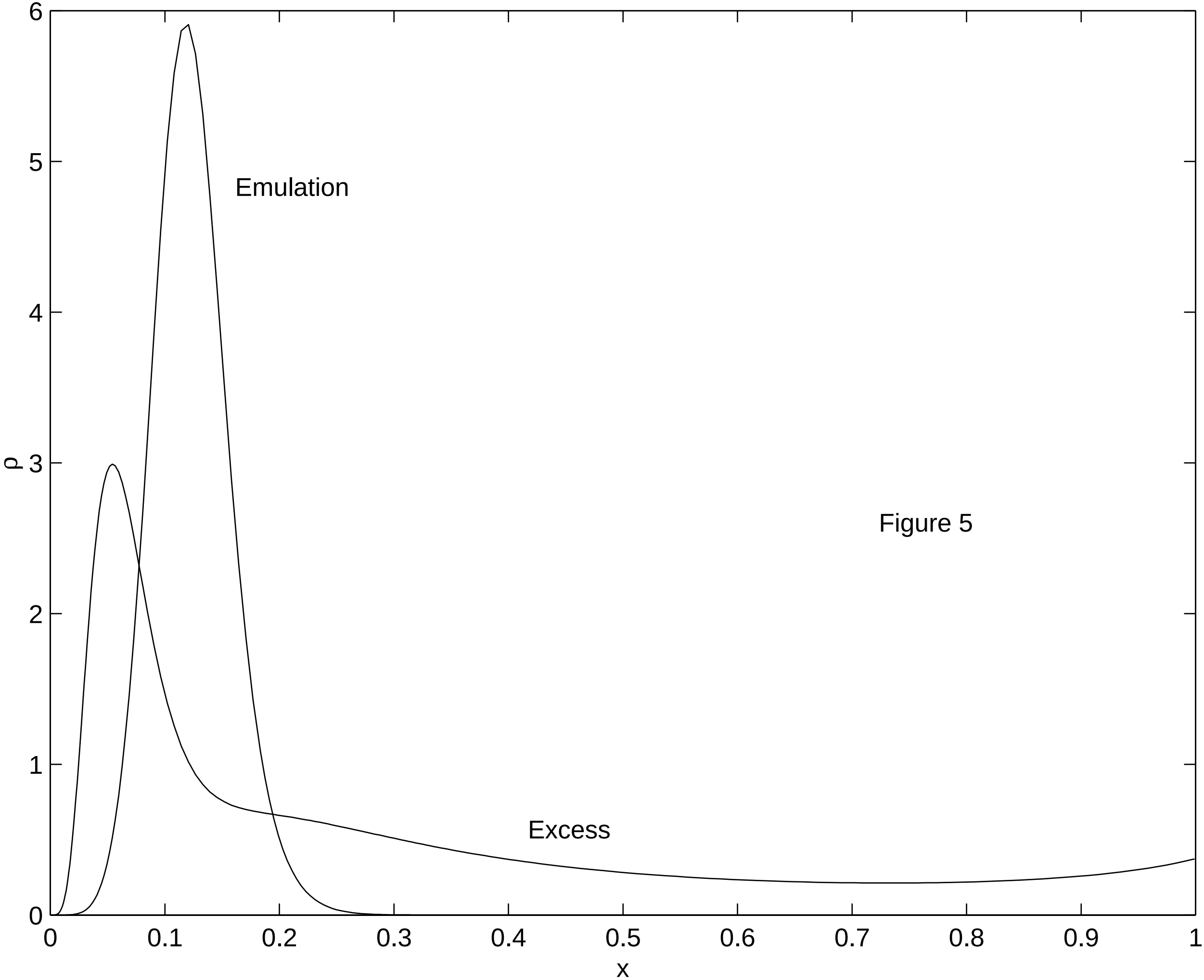

t=.75
t=.50
t=.25
Figure 1
ρ
x
0
2
4
6
8
10
12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


D*(x)
D(x,0)
Figure 2
D(x)
x
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


D*(x)
D(x,0)
Figure 3
D(x)
x
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


0
0.1



Emulation
Excess
Figure 5
ρ
x
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1